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Zeros of Generalized Krawtchouk Polynomials*

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The zeros of generalized Krawtchouk polynomials are studied. Some interlacing theorems for the zeros are given. A new infinite family of integral zeros is given, and it is conjectured that these comprise most of the non-trivial zeros. The integral zeros for two families of q-Krawtchouk polynomials are classified.

1. INTRODUCTION

Given a sequence of orthogonal polynomials $p_n(x)$, it is well known [5, p. 27] that the zeros of $p_n(x)$ are real, simple, and lie inside the interval of orthogonality. The Krawtchouk polynomials are a finite sequence of orthogonal polynomials $p_n(x)$, $0 \leqslant n \leqslant N$, whose interval of orthogonality is $[0, N]$. In this paper we shall consider the zeros of Krawtchouk polynomials, and in particular investigate the integral zeros.

The integrality of zeros for orthogonal polynomials has combinatorial importance. If the polynomials are naturally related to an association scheme, then the location of the zeros is critical for combinatorial properties of the scheme. For example, if the scheme has a configuration called a perfect e-code, then the polynomial of degree e has e integral zeros [7, Chap. 5]. As another example, generalized Radon transforms can be

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defined on association schemes [9]. Such a transform is invertible if and only if the relevant polynomial does not have integral zeros.

The most important association scheme is the Hamming scheme of classical coding theory [3], [12], and its polynomials are the Krawtchouk polynomials. Thus, the zeros of these polynomials and their q-analogs [8] are important. We give some elementary interlacing properties of the zeros in Section 3. A new infinite family of integral zeros is given in Theorem 4.6. Numerical evidence supports Conjecture 4.7 that asymptotically these are most of the non-trivial integral zeros. In Theorem 6.1 we show that a family of q-Krawtchouk polynomials never has integral zeros. Finally, in Section 6 we consider a second family of q-Krawtchouk polynomials, and classify its integral zeros.

For the (generalized) Krawtchouk polynomials, we need to introduce the standard notation for (basic) hypergeometric series,

$$
{}_{r+1}F_r\begin{pmatrix} a_1 & a_2 & \cdots & a_{r+1}; & x \\ b_1 & b_2 & \cdots & b_r & \end{pmatrix} = \sum_{j=0}^{\infty} \frac{(a_1)_j (a_2)_j \cdots (a_{r+1})_j x^j}{(b_1)_j (b_2)_j \cdots (b_r)_j \, j!}, \tag{1.1}
$$

where

$$
(a)_j = a(a+1)\cdots(a+j-1), \tag{1.2}
$$

and

$$
{}_{r+1}\phi_r\begin{pmatrix} a_1 & a_2 & \cdots & a_{r+1}; & q, & x \\ b_1 & b_2 & \cdots & b_r & & \end{pmatrix} = \sum_{j=0}^{\infty} \frac{(a_1)_j (a_2)_j \cdots (a_{r+1})_j x^j}{(b_1)_j (b_2)_j \cdots (b_r)_j (q)_j}, \tag{1.3}
$$

where

$$
(a)_j = (a; q)_j = (1-a)(1-aq)\cdots(1-aq^{j-1}). \tag{1.4}
$$

2. Preliminaries

In this section we review the necessary facts about the Krawtchouk polynomials. We refer the reader to [2] or [11] for more details.

We shall use notation for Krawtchouk polynomials which agrees with the Hamming scheme $H(N, q)$. The Krawtchouk polynomial $k_n(x, q, N)$ of degree n in x is orthogonal on $x = 0, 1, \ldots, N$ with respect to the measure

$\binom{N}{x}(q-1)^x$. We may take $q>1$, although q is integral for $H(N, q)$. This defines the polynomials up to a normalization constant. They may also be defined by the three-term recurrence relation

$$\begin{aligned}(n+1)\, k_{n+1}(x) \\ &= [(N-n)(q-1)+n-qx]\, k_n(x)-(q-1)(N-n+1)\, k_{n-1}(x) \qquad (2.1)\end{aligned}$$

for $n=1, ..., N-1$, with the initial conditions $k_0(x)=1$ and $k_1(x)=-qx+N(q-1)$. Their generating function is

$$\sum_{n=0}^{N} k_n(x, q, N)\, z^n=(1+(q-1)z)^{N-x}\,(1-z)^x. \qquad (2.2)$$

From (2.2), it is easy to see

$$k_n(x, q, N)=\sum_{j=0}^{n} (-1)^j\,(q-1)^{n-j}\binom{N-x}{n-j}\binom{x}{j} \qquad (2.3)$$

and thus deduce

$$k_n(0, q, N)=\binom{N}{n}(q-1)^n. \qquad (2.4)$$

Also

$$k_n(x, q, N)=k_n(N-x, q/(q-1), N)(1-q)^n. \qquad (2.5)$$

The Krawtchouk polynomials can also be expressed as a ${}_2F_1$:

$$k_n(x, q, N)=\binom{N}{n}(q-1)^n\,{}_2F_1\left(\begin{matrix}-n & -x; \\ & -N\end{matrix}\;\frac{q}{q-1}\right). \qquad (2.6)$$

Because a ${}_2F_1$ is a symmetric function of the two numerator parameters, if $x=i$ is integral we have the self-dual relation

$$\begin{aligned}\binom{N}{i}(q-1)^i\, k_n(i, q, N) \\ &=\binom{N}{n}(q-1)^n\, k_i(n, q, N), \qquad n, i=0, 1, ..., N. \qquad (2.7)\end{aligned}$$

Clearly the three-term relation (2.1) and the self-dual relation (2.7) imply the difference equation

$$\begin{aligned}&(q-1)(N-x)\, k_n(x+1, q, N)-[(q-1)(N-x)+x-qn] \\ &\quad\times k_n(x, q, N)+xk_n(x-1, q, N)=0. \qquad (2.8)\end{aligned}$$

In later sections we will need the following propositions. Proposition 2.1 follows from the generating function (2.2), and Proposition 2.2 follows from parts (1) and (2) of Proposition 2.1.

PROPOSITION 2.1. *For* $0 \leqslant n \leqslant N$,

(1) $k_n(x, q, N+1) = k_n(x, q, N) + (q-1)\, k_{n-1}(x, q, N)$,

(2) $k_n(x, q, N+1) = k_n(x-1, q, N) - k_{n-1}(x-1, q, N)$,

(3) $k_n(x, q, N) - k_n(x-1, q, N) + (q-1)\, k_{n-1}(x, q, N) + k_{n-1}(x-1, q, N) = 0$,

(4) $k_n(x, q, N) - k_n(x+1, q, N) = q k_{n-1}(x, q, N-1)$.

PROPOSITION 2.2. *If* $k_r(s, q, N) = 0$, *where* s *is real, then*

(1) $k_r(s, q, N+1) = (q-1)\, k_{r-1}(s, q, N)$,

(2) $k_{r+1}(s, q, N+1) = k_{r+1}(s, q, N)$,

(3) $k_r(s+1, q, N+1) = -k_{r-1}(s, q, N)$.

Finally, we need a special proposition for the $q = 2$ case.

PROPOSITION 2.3. *We have*

$$k_r(x, 2, N) - k_r(x+2, 2, N) = 4k_{r-1}(x, 2, N-2).$$

Proof. From Proposition 2.1(4) we obtain the two equations

$$k_r(x, 2, N) - k_r(x+1, 2, N) = 2k_{r-1}(x, 2, N-1)$$

$$k_r(x+1, 2, N) - k_r(x+2, 2, N) = 2k_{r-1}(x+1, 2, N-1).$$

Adding these two equations together yields

$$\begin{aligned} &k_r(x, 2, N) - k_r(x+2, 2, N) \\ &\quad = 2[k_{r-1}(x, 2, N-1) + k_{r-1}(x+1, 2, N-1)] \\ &\quad = 2[-k_r(x+1, 2, N-1) + k_r(x, 2, N-1)] \\ &\quad = 2[2k_{r-1}(x, 2, N-2)], \end{aligned}$$

where the second equality follows from Proposition 2.1(3) and the third equality follows from Proposition 2.1(4). ∎

3. INTERLACING OF THE ZEROS

In this section we concentrate on properties of the zeros which do not involve integrality. Let $x_{n,1}^N < x_{n,2}^N < \cdots < x_{n,n}^N$ denote the zeros of

$k_n(x, q, N)$. Recall that $0 < x_{n,1}^N < x_{n,n}^N < N$, there is some integer in any open interval $(x_{n,k}^N, x_{n,k+1}^N)$ [16], and that the zeros of $k_n(x, q, N)$ and $k_{n-1}(x, q, N)$ interlace [16]. First we give an interlacing theorem which is analogous to [10, Theorem 4] for Hahn polynomials.

THEOREM 3.1. *The zeros of $k_n(x, q, N)$ and $k_n(x, q, N+1)$ interlace,*

$$x_{n,i}^N < x_{n,i}^{N+1} < x_{n,i+1}^N < x_{n,i+1}^{N+1}, \qquad \textit{for} \quad i = 1, ..., n-1.$$

Proof. This follows from the interlacing property for $k_n(x, q, N)$ and $k_{n-1}(x, q, N)$ and Proposition 2.1(1). The details are similar to the proof of Theorem 3.2. ∎

We now have two zeros, $x_{n,i}^{N+1}$ and $x_{n-1,i}^N$, which lie in the interval $[x_{n,i}^N, x_{n,i+1}^N]$. These zeros also interlace.

THEOREM 3.2. *The zeros of $k_{n-1}(x, q, N)$ and $k_n(x, q, N+1)$ interlace,*

$$x_{n,i}^{N+1} < x_{n-1,i}^N < x_{n,i+1}^{N+1}, \qquad \textit{for} \quad i = 1, ..., n-1.$$

Proof. We must show that $x_{n,i}^{N+1} < x_{n-1,i}^N$, so we first consider the $i = 1$ case. Clearly the interlacing property for $k_n(x, q, N)$ and $k_{n-1}(x, q, N)$ and Proposition 2.1(1) imply that $x_{n,1}^{N+1} \neq x_{n-1,1}^N$. So assume that $x_{n-1,1}^N < x_{n,1}^{N+1}$. Because the leading terms of $k_n(x, q, N)$ and $k_n(x, q, N+1)$ have the same sign, $k_n(x, q, N)$ and $k_n(x, q, N+1)$ have opposite signs at $x_{n-1,1}^N$. This contradicts Proposition 2.2(2).

The proof continues by induction on i noting that $k_n(x, q, N)$ and $k_n(x, q, N+1)$ have opposite signs on the interval $(x_{n,i}^N, x_{n,i}^{N+1})$. So the assumption $x_{n-1,i}^N < x_{n,i}^{N+1}$ would contradict Proposition 2.2(2). ∎

The next theorem shows that the interlacing in Theorem 3.1 is "close."

THEOREM 3.3. *For $1 \leqslant i \leqslant n \leqslant N$, $x_{n,i}^N - x_{n,i}^{N+1} < 1$.*

Proof. Since $k_n(x_{n,i}^N, q, N) = 0$, Proposition 2.2(1) implies

$$k_n(x_{n,i}^N, q, N+1) = (q-1)\, k_{n-1}(x_{n,i}^N, q, N)$$

and Proposition 2.2(2) implies

$$k_n(x_{n,i}^N + 1, q, N+1) = -k_{n-1}(x_{n,i}^N, q, N).$$

Thus, we have

$$k_n(x_{n,i}^N, q, N+1) = -(q-1)\, k_n(x_{n,i}^N + 1, q, N+1),$$

and if $q>1$, then $k_n(x, q, N+1)$ must have a zero between $x_{n,i}^N$ and $x_{n,i}^N+1$. ∎

If $q=2$, we can say more about the distance between consecutive zeros. Levit [10, Theorem 3] has a similar result for Hahn polynomials.

THEOREM 3.4. *Let* $x_1 < x_2$ *be consecutive zeros of* $k_r(x, 2, N)$ *and suppose* $r < N/2$. *Then*

$$x_1 + 2 < x_2.$$

Proof. First we note that the difference equation (2.8) when $q=2$ is

$$(N-2r)\, k_r(x) = (N-x)\, k_r(x+1) + x k_r(x-1). \tag{3.1}$$

Assume that x_1 is not an integer (the argument for x_1 an integer is similar). Suppose there is only one integer s separating x_1 and x_2, that is, $s-1 \leqslant x_1 < s < x_2 \leqslant s+1$. (Recall that the open interval between two consecutive zeros of an orthogonal polynomial must contain at least one spectral point.) Then we must have

$$\operatorname{sgn}(k_r(s-1)) = \operatorname{sgn}(k_r(s+1)) = -\operatorname{sgn}(k_r(s)),$$

which contradicts (3.1) if $x=s$. This contradiction forces there to be at least two integers in the interval (x_1, x_2), or $x_2 > x_1 + 1$.

Now suppose $x_2 \leqslant x_1 + 2$, and let $x = x_1 + 1$ in (3.1). Then

$$(N-2r)\, k_r(x_1+1) = (N - x_1 - 1)\, k_r(x_1+2),$$

which implies

$$\operatorname{sgn}(k_r(x_1+1)) = \operatorname{sgn}(k_r(x_1+2)). \tag{3.2}$$

Then (3.2) and $x_1 + 1 < x_2 \leqslant x_1 + 2$ imply that there must be another zero of $k_r(x)$ between x_2 and x_1+2. But from the previous paragraph, we know then that (x_2, x_1+2) must contain at least two integers, thus giving us four integers in the interval (x_1, x_1+2). This is clearly impossible; thus $x_2 > x_1 + 2$. ∎

The condition $r < N/2$ in Theorem 3.4 cannot be relaxed because the zeros of $k_N(x, 2, 2N)$ are $1, 3, ..., 2N-1$.

There is a discrete form of Markoff's theorem [16, p. 115] which states the following. If the weight function $w(x, q)$ is purely discrete, and the logarithmic derivative $w_q(x, q)/w(x, q)$ is increasing, then the ith zero of the corresponding orthogonal polynomials is an increasing function of q. For the Krawtchouk polynomials we then have the following theorem.

THEOREM 3.5. *Let $x_{i,n}^N(q)$ be the* ith *zero of $k_n(x, q, N)$. Then $x_{i,n}^N(q)$ is an increasing function of q, for $q > 1$.*

4. INTEGRAL ZEROS

We next consider the number theoretic conditions which are necessary for a Krawtchouk polynomial to have an integral zero. We also give a non-trivial family of integral zeros in Theorem 4.6.

THEOREM 4.1. *If $k_r(s, q, N) = 0$, s an integer, then q divides $\binom{N}{r}$.*

Proof. From Proposition 2.1(4), we have

$$k_r(0, q, N) - k_r(1, q, N) = qk_{r-1}(0, q, N-1)$$
$$k_r(1, q, N) - k_r(2, q, N) = qk_{r-1}(1, q, N-1)$$
$$\vdots$$
$$k_r(s-1, q, N) - k_r(s, q, N) = qk_{r-1}(s-1, q, N-1).$$

The telescoping sum yields

$$k_r(0, q, N) = q \sum_{j=0}^{s-1} k_r(j, q, N-1).$$

Recalling (2.4) yields the desired conclusion, since $k_r(j, q, N-1)$ is integral. ∎

If q is a prime number, the value of $\binom{N}{r} \pmod q$ is well known from Lucas' theorem [6, p. 65], and we have the following corollary.

COROLLARY 4.2. *Let $r_1 \cdots r_a$ and $N_1 \cdots N_b$ be the base q representations of r and N, respectively. If q is a prime number and $k_r(s, q, N) = 0$ for some integer s, then for some i, $r_i > N_i$.*

The divisibility condition in Theorem 4.1 can be improved if $q = 2$.

THEOREM 4.3. *If s is a positive integer and $k_r(s, 2, N) = 0$, then*

$$k_r(s(\text{mod } 2^t), 2, N) \equiv 0 \qquad (\text{mod } 2^{t+1}),$$

for any integer $t \geq 1$.

Proof. Proposition 2.3 immediately gives the result for $t=1$. It also shows

$$k_r(x, 2, N) - k_r(x+4, 2, N)$$
$$= 4[k_{r-1}(x, 2, N-2) + k_{r-1}(x+2, 2, N-2)]. \tag{4.1}$$

If x is integral, then Proposition 2.3 implies that $k_{r-1}(x, 2, N-2)$ and $k_{r-1}(x+2, 2, N-2)$ are the same modulo four, so the left side of (4.1) $\equiv 0$ (mod 8). Again a telescoping sum shows that this is the $t=2$ case. The general case follows from $k_r(x, 2, N) - k_r(x+2^t, 2, N)$ being the sum of 2^{t-1} terms, all of which are the same modulo four. ∎

We next consider the integral zeros of $k_n(x, 2, N)$. By symmetry in n and x we can assume that $n \leqslant x$. From (2.5), we can clearly take $x \leqslant N/2$. There is another set of zeros to which we will refer as trivial. For n odd, (2.5) also implies that $x = N/2$ is an integral zero for N even.

For polynomials of small degree n, the non-trivial integral zeros (n, x, N), $1 \leqslant n \leqslant x \leqslant N/2$, can be found explicitly (see [9]).

PROPOSITION 4.4. *The integral zeros for degrees* 1, 2, *and* 3 *are*

(1) $(1, k, 2k)$, $k \geqslant 1$,

(2) $(2, k(k-1)/2, k^2)$, $k \geqslant 3$,

(3) $(3, k(3k \pm 1)/2, 3k^2 + 3k + 3/2 \pm (k + 1/2))$, $k \geqslant 2$.

Strictly speaking, (1) consists of trivial zeros. Note that the pentagonal numbers appear as the integral zeros in (3).

Even though the zeros can be given for degree four, it is a difficult number theoretic problem to classify when these zeros are integral. Graham and Diaconis [9] give nontrivial values of (4, 7, 17), (4, 10, 17), (4, 30, 66), (4, 36, 66), (4, 715, 1521), (4, 806, 1521), (4, 7476, 15043), (4, 7567, 15043), along with the trivial values (4, 1, 8), (4, 3, 8), (4, 5, 8), (4, 7, 8). They conjecture that this is the complete list for degree four. Laurent Habsieger has shown that any other possible value of N must have at least 1000 digits. We can use Theorem 4.3 to restrict the possible values of N. The following corollary takes $t=2$.

COROLLARY 4.5. *Suppose* $k_4(x, 2, N) = 0$, *for some integer* x.

(1) *If* $x \equiv 0$ (mod 4), *then* $N \equiv 0, 1, 2,$ *or* 3 (mod 32).

(2) *If* $x \equiv 1$ (mod 4), *then* $N \equiv 1, 2, 3,$ *or* 8 (mod 32).

(3) *If* $x \equiv 2$ (mod 4), *then* $N \equiv 0, 2, 3,$ *or* 17 (mod 32).

(4) *If* $x \equiv 3$ (mod 4), *then* $N \equiv 2, 3, 8,$ *or* 17 (mod 32).

Proof. Use the explicit form of $k_4(i, 2, N) = 0$, $0 \leqslant i \leqslant 3$, as a polynomial of degree i in N and Theorem 4.3. ∎

Using MACSYMA we found all of the non-trivial integral zeros for $q = 2$ and $N \leqslant 700$. The data suggested the following theorem, for an infinite family of non-trivial zeros.

THEOREM 4.6. *For any integer $h \geqslant 1$,*

$$k_{2h}(4h-1, 2, 8h+1) = 0.$$

Proof. This theorem can be proved using the theory of hypergeometric series in [1]. Instead we give a simple proof from the $q = 1$ version of (6.6) below. Let $N = 2n + t$ in (6.6), and assume that x is even. If $t = 3$ the two allowed values of r in (6.6) are $r = x/2$ and $r = x/2 - 1$. If these two terms sum to zero, we find that $x = (n+1)/2 = 2h$, so that $n = 4h - 1$ and $N = 8h + 1$. ∎

The following table lists all non-trivial integral zeros for $q = 2$, $N \leqslant 700$, and $1 \leqslant n \leqslant x \leqslant N/2$, which do not follow from Proposition 4.4 or Theorem 4.6.

(5, 14, 36)	(23, 31, 67),	(31, 103, 214)	(34, 254, 514)
(5, 22, 67)	(14, 47, 98)	(5, 133, 289)	(84, 286, 576)
(5, 28, 67)	(19, 62, 132)	(6, 155, 345)	
(6, 31, 67)	(61, 86, 177)	(44, 230, 465)	

Note that many of these zeros have $N = 2n + t$, for small values of t. In fact, with Laurent Habsieger we have found five more infinite families of zeros, for $t = 4, 5, 6$, and 8. These families contain all of the zeros in the table, except for (5, 22, 67), (5, 28, 67), (5, 133, 289), and (6, 155, 345). The details will appear elsewhere. Nevertheless, these zeros are more sparse than those in Theorem 4.6, and we make the following conjecture.

Conjecture 4.7. The number of non-trivial integral zeros of Krawtchouk polynomials $k_n(x, 2, M)$ for $M \leqslant N$ is asymptotic to $N/8$.

One may also look for families of integral zeros of $k_n(x, q, N)$, $q \geqslant 3$. In this case we may assume $1 \leqslant n \leqslant \min(x, N - x)$. Again using MACSYMA this was done for $q \leqslant 20$ and $N \leqslant 100$.

PROPOSITION 4.8. *The integral zeros (n, x, N) of $k_n(x, q, N)$ for $n \geqslant 3$, $N \leqslant 100$, and $3 \leqslant q \leqslant 20$ are*

(1) (3, 14, 28), (5, 16, 30), (4, 35, 57), (4, 65, 93), *for* $q=3$,

(2) (3, 55, 66), (5, 55, 68), *for* $q=4$,

(3) (4, 66, 80), *for* $q=5$, *and*

(4) (4, 52, 66), *for* $q=6$.

5. Summation Theorems

In this section we give a few summation theorems for Krawtchouk polynomials which follow from the recurrence relations in Section 2.

From Proposition 2.1(4) we can deduce the following identity for the Krawtchouk polynomials:

$$\sum_{j=0}^{N} k_r(j, q, N) = \frac{1}{q}\left[(q-1)^{r+1} + (-1)^r\right]\binom{N+1}{r+1}. \tag{5.1}$$

This can be generalized for $q=2$ by using Lemma 4.3 to find if N is even and r odd,

$$\sum_{j=0}^{N/2} k_r(2j, 2, N) = 0 = \sum_{j=1}^{N/2} k_r(2j-1, 2, N), \tag{5.2}$$

if N and r are both even,

$$\begin{aligned} \sum_{j=0}^{N/2} k_r(2j, 2, N) &= \frac{N+2}{2(N-r+1)}\binom{N+1}{r+1} \\ \sum_{j=1}^{N/2} k_r(2j-1, 2, N) &= \frac{N-2r}{2(N-r+1)}\binom{N+1}{r+1}, \end{aligned} \tag{5.3}$$

if N and r are both odd,

$$\sum_{j=0}^{(N-1)/2} k_r(2j, 2, N) = \frac{1}{2}\binom{N+1}{r} = -\sum_{j=1}^{(N+1)/2} k_r(2j-1, 2, N), \tag{5.4}$$

and if N is odd and r is even,

$$\sum_{j=0}^{(N-1)/2} k_r(2j, 2, N) = \frac{1}{2}\binom{N+1}{r+1} = \sum_{j=1}^{(N+1)/2} k_r(2j-1, 2, N). \tag{5.5}$$

From (5.1) and the product formula for Krawtchouk polynomials it is easy to see that

$$\sum_{k=0}^{N} k_r(k, 2, N)\, k_s(k, 2, N)$$
$$= \sum_{m=0}^{N/2} \binom{N-2m}{(r+s)/2-m} \binom{2m}{(r-s)/2-m} \binom{N+1}{2m+1}. \tag{5.6}$$

The binomial coefficients in (5.6) which have non-integral arguments are defined to be zero.

6. q-ANALOGS

There are two families of orthogonal polynomials which have been called q-Krawtchouk polynomials. In this section we shall consider the zeros of these polynomials, and compare the results to Section 4.

The first family has been called the *affine q-Krawtchouk* polynomials, because it arises from association schemes with natural translations [14, 15]

$$K_n^{\mathrm{Aff}}(x, a, N, q) = v_n\, {}_3\phi_2\left(\begin{matrix} q^{-x} & q^{-n} & 0; & q, & q \\ & q^{-N} & a & & \end{matrix}\right), \qquad \text{for} \quad 0 \leqslant n \leqslant N, \tag{6.1}$$

where v_n is non-zero. Clearly (6.1) implies that $K_n^{\mathrm{Aff}}(x, a, N, q)$ is a polynomial in q^{-x} of degree n. The three cases for which these polynomials are realized from association schemes all have q equal to a prime power. They are

(A1) $K_n^{\mathrm{Aff}}(x, q^{-M}, N, q)$, M integral and $N \leqslant M$,

(A2) $K_n^{\mathrm{Aff}}(x, q^{2\lfloor (N+1)/2 \rfloor -1}, \lfloor N/2 \rfloor, q^2)$, and

(A3) $K_n^{\mathrm{Aff}}(x, -(-q)^{-N}, N, -q)$.

Note that the effect of the greatest integer functions in (A2) is just to put q^{-N} and q^{1-N} as denominator parameters in the ${}_3\phi_2$. Note also that if v_n is defined appropriately,

$$\lim_{q \to 1} K_n^{\mathrm{Aff}}(x, a, N, q) = k_n(x, 1/a, N).$$

However, this limit does not apply to the three cases above, because the value of a depends upon q.

The other family of q-Krawtchouk polynomials is a q-analog of the *binary* Krawtchouk polynomials (the $q=2$ case in Section 2). It is

$$K_n(x, c, N, q)$$
$$= v_n {}_3\phi_2\begin{pmatrix} q^{-x} & q^{-n} & -q^{n-N-c}; & q, & q \\ & q^{-N} & 0 & & \end{pmatrix}, \qquad \text{for} \quad 0 \leqslant n \leqslant N, \tag{6.2}$$

where v_n is non-zero. These polynomials arise from six association schemes [13], giving five sets of polynomials:

(B1) $K_n(x, 0, N, q)$,

(B2) $K_n(x, 1, N, q)$,

(B3) $K_n(x, 2, N, q)$,

(B4) $K_n(x, 1/2, N, q^2)$,

(B5) $K_n(x, 3/2, N, q^2)$.

This time

$$\lim_{q \to 1} K_n(x, c, N, q) = k_n(x, 2, N)$$

if v_n is defined appropriately. There are also group theoretic reasons [14] for considering these polynomials as the correct q-analogs of $k_n(x, 2, N)$.

First we consider the affine q-Krawtchouk polynomials (A1)–(A3). Surprisingly, the next theorem states that these polynomials are never zero at integral values of x. It had previously been shown [4] that at least one zero must be non-integral.

THEOREM 6.1. *Let $p_n(x)$ be one of the affine q-Krawtchouk polynomials* (A1)–(A3), *where q is a prime power. Then*

(1) $p_n(i) \neq 0$ *for* $i, n = 0, 1, ..., N$, *for* (1) *and* (3),

(2) $p_n(i) \neq 0$ *for* $i, n = 0, 1, ..., \lfloor N/2 \rfloor$, *for* (2).

Proof. We shall prove case (2), and leave the other two cases to the reader. By symmetry we can assume that $i \geqslant n \geqslant 1$ and $p_n(i) = 0$. Multiply the definition (6.1) of $p_n(i)$ by $(q^{-N}; q^2)_n (q^{-N+1}; q^2)_n (q^2; q^2)_n q^{2nN}$ to obtain

$$\sum_{j=0}^{n} (q^{2n}; q^{-2})_j (q^{2i}; q^{-2})_j (q^{N-2j}; q^{-2})_{n-j} (q^{N-1-2j}; q^{-2})_{n-j}$$
$$\times (q^{2j+2}; q^2)_{n-j} q^{(N-2n)j+(N-2i)j+2n^2-n+j} = \sum_{j=0}^{n} c_j(q) = 0. \tag{6.3}$$

For each j, $c_j(q)$ is a polynomial in q with integer coefficients. The lowest power of q that appears in $c_j(q)$ is $(N-2n)\,j+(N-2i)\,j+2n^2-n+j$ and the highest power of q that appears is $2Nn+n(n+1)-j^2+j$. Since $i, j \leqslant \lfloor N/2 \rfloor$, (6.3) implies that the lowest power of q appearing is q^{2n^2-n}. Thus, if we divide both sides by q^{2n^2-n}, we obtain $0 \equiv 1 \pmod q$. This contradiction implies that $p_n(i) \neq 0$ for $i \geqslant n$. ∎

Next we turn to the q-Krawtchouk polynomials $K_n(x, c, N, q)$. These polynomials are not symmetric in n and x. We shall need the generating function [13]

$$\sum_{x=0}^{N} K_n(x, c, N, q)\, z^x = (z)_n(-q^c z)_{N-n}, \tag{6.4}$$

which implies the following q-analog of (2.3) via the q-binomial theorem [1]:

$$K_n(x, c, N, q) = \sum_{j=0}^{x} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} N-n \\ x-j \end{bmatrix}_q q^{\binom{x}{2}+(c-j)(x-j)}. \tag{6.5}$$

We first give the q-analog of the trivial zeros of x odd for $k_N(x, 2, 2N)$.

PROPOSITION 6.2. *If x is an odd integer satisfying $1 \leqslant x \leqslant 2N$, then $K_N(x, 0, 2N, q) = 0$.*

Proof. This is clear from (6.4) and $(z)_N(-z)_N = (z^2; q^2)_N$. ∎

Finally, the argument of Theorem 6.1 implies that these polynomials are otherwise non-zero.

THEOREM 6.3. *If q is a prime power, the following values of q-Krawtchouk polynomials are non-zero:*

(1) $K_n(i, 0, N, q)$ *for* $0 \leqslant i,\ n \leqslant N,\ 2n \neq N$,

(2) $K_n(i, 1, N, q)$ *for* $0 \leqslant i,\ n \leqslant N$,

(3) $K_n(i, 2, N, q)$ *for* $0 \leqslant i,\ n \leqslant N$,

(4) $K_n(i, 1/2, N, q^2)$ *for* $0 \leqslant i,\ n \leqslant N$,

(5) $K_n(i, 3/2, N, q^2)$ *for* $0 \leqslant i,\ n \leqslant N$.

Proof. First we take $c = 0$ and assume that $2n < N$, so that the generating function is $(z^2; q^2)_n(-zq^n; q)_{N-2n}$. Then the q-binomial theorem [1] implies

$$\begin{aligned} K_n(x, 0, N, q) &= \sum_r \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} q^{2\binom{r}{2}} \begin{bmatrix} N-2n \\ x-2r \end{bmatrix}_q q^{\binom{x-2r}{2}+n(x-2r)}(-1)^r \\ &= \sum_r c_r(q), \end{aligned} \tag{6.6}$$

where $c_r(q)$ is a polynomial in q with integral coefficients. The degree of the lowest-degree term in $c_r(q)$ is

$$L(r) = nx + x^2/2 - x/2 + 3r^2 - 2rx - 2rn.$$

As a function of r, $L(r)$ has its minimum value at $r = (x+n)/3$. Thus there is a unique term in (6.6) containing the smallest power of q, and as in Theorem 6.1, this implies $K_n(x, 0, N, q) \neq 0$. The $2n > N$ case can be done similarly.

The proof for $c = 1$ ($c = 2$) is similar, this time two (three) terms naturally occur in the sum that corresponds to (6.6). Nevertheless, again there is a unique term with the term of minimum degree.

For $c = \frac{1}{2}$ or $c = \frac{3}{2}$, we can use (6.5), for which

$$L(r) = x^2/2 - x/2 + r^2 - rx - rc + cx.$$

The minimum value for $L(r)$ occurs at $r = (x+c)/2$. If $c = \frac{1}{2}$ or $c = \frac{3}{2}$, this insures a unique term in (6.5) of minimum degree. Note that this argument also does the $c = 0$ case with x even, but not for x odd. ∎

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